TEMPERATURE FIELDS IN LAYERED MEDIA WITH MOVING BOUNDARIES

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The one-dimensional problem of nonlinear heat conduction in a multilayered system is considered taking account of mass entrainment. In the general case, the rate of mass entrainment is assumed an arbitrary function of the time, the temperature, the heat flux, and the coordinates of the moving boundaries of the system.

1. Formulation of the Problem

The temperature distribution in the layers $x_{j-1} < x < x_j$ of a multilayer system is described by the equations

$$c_j\rho_j \frac{\partial T_j}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_j \frac{\partial T_j}{\partial x}\right),\ t\in(t_0,\ t_1],\ 1 \leqslant j \leqslant n \tag{1}$$

with the boundary conditions

$$F_0\left[t,\ T(x_0),\ \frac{\partial T}{\partial x}(x_0)\right] = 0, \quad F_n\left[t,\ T(x_n),\ \frac{\partial T}{\partial x}(x_n),\ \omega\right] = 0, \tag{2}$$

the conditions of discontinuous T and $\partial T/\partial x$ at the points x_j

$$F_j^{(i)}\left[t,\ T(x_j\pm),\ \frac{\partial T}{\partial x}(x_j\pm)\right] = 0, \quad i = 1,\ 2; \quad 1 \leqslant j \leqslant n-1 \tag{3}$$

and the initial distribution

$$T_j(t_0,\ x) = T_j^0(x), \quad 1 \leqslant j \leqslant n. \tag{4}$$

It is assumed that the dynamics of mass entrainment (accretion) on the outer surface of the layer $x_{n-1} < x < x_n$ is described by a differential equation of the form

$$\frac{dx_n}{dt} = \omega\left[t,\ x_n,\ T(x_n),\ \frac{\partial T}{\partial x}(x_n)\right] \tag{5}$$

with the initial condition

$$x_n(t_0) = x_n^0.$$

In particular, (5) describes the motion of the boundary $x = x_n(t)$ in processes on the surface such as sublimation, condensation, mass entrainment during mechanical treatment, etc. Equations (1) and (5), together with conditions (2)-(4) and (6), completely determine the temperature in the layers and the value of $x_n(t)$ for $t > t_0$. The boundary conditions (2) and the condition of thermal contact of the layers (3) are nonlinear in the general case, the thermophysical properties of the layer material (c, ρ, λ) can depend on x and T. The dependence of c and λ on T can exert considerable influence on the temperature distribution in the neighborhood of the point $x_n(t)$.

It is convenient to reduce the problem under consideration to a problem with fixed boundaries by introducing the new independent variable

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$$\bar{x} = x, \quad x_0 \leqslant x \leqslant x_{n-1},$$

$$\bar{x} = x_{n-1} + \frac{x - x_{n-1}}{x_n(t) - x_{n-1}} (x_n^0 - x_{n-1}), \quad x_{n-1} < x \leqslant x_n(t)$$

for points of the outer layer.

Let us also introduce dimensionless variables by means of the formulas

$$T_j = u_j T_*, \quad \bar{x} = \xi x_*, \quad x = \tilde{x} x_*, \quad t = \tau t_*, \quad c_j = \delta_j c_*, \quad \rho_j = \gamma_j \rho_*, \quad \lambda_j = k_j \lambda_*.$$

Here T_*, x_*, t_*, c_*, ρ_* and λ_* are the characteristic dimensions of the temperature, length, time, specific heat, density, and heat conductivity, respectively. The relationships (1)-(6) become in the new variables

$$\frac{\partial u_j}{\partial \tau} = Q_j^{(1)} \frac{\partial^2 u_j}{\partial \xi^2} + Q_j^{(2)}, \quad \tau \in (\tau_0, \tau_1], \quad \xi \in (\xi_{j-1}, \xi_j), \quad 1 \leqslant j \leqslant n, \tag{1'}$$

$$\Phi_0 \left[\tau, \ u(\xi_0), \ \frac{\partial u}{\partial \xi}(\xi_0)\right] = 0, \quad \Phi_n \left[\tau, \quad u(\xi_n), \ \frac{\partial u}{\partial \xi}(\xi_n), \ \bar{\omega}\right] = 0, \tag{2'}$$

$$\Phi_j^{(i)} \left[\tau, \ u(\xi_j \pm), \ \frac{\partial u}{\partial \xi}(\xi_j \pm)\right] = 0, \quad i = 1, \ 2; \quad 1 \leqslant j \leqslant n-1, \tag{3'}$$

$$\frac{d\eta}{d\tau} = \bar{\omega}\left[\tau, \ \eta, \ u(\xi_n), \ \frac{\partial u}{\partial \xi}(\xi_n)\right],$$

$$u_j(\tau_0, \xi) = u_j^0(\xi), \quad \eta(\tau_0) = \xi_n.$$

The coefficients $Q_j^{(1)}$, $Q_j^{(2)}$ are defined by the equalities

$$Q_j^{(1)} = \frac{a^2 k_j}{\gamma_j \delta_j}, \quad Q_j^{(2)} = \frac{a^2}{\gamma_j \delta_j}\left(\frac{\partial k_j}{\partial u_j} \cdot \frac{\partial u_j}{\partial \xi} + \frac{\partial k_j}{\partial \xi}\right)\frac{\partial u_j}{\partial \xi}, \quad 1 \leqslant j \leqslant n-1,$$

$$Q_n^{(2)} = \left[\frac{a^2}{\gamma_n \delta_n}\left(\frac{\partial k_n}{\partial u} \cdot \frac{\partial u_n}{\partial \xi} + \frac{\partial k_n}{\partial \xi}\right)\left(\frac{\xi_n - \xi_{n-1}}{\eta - \xi_{n-1}}\right)^2 + \bar{\omega}\,\frac{\xi - \xi_{n-1}}{\eta - \xi_{n-1}}\right]\frac{\partial u_n}{\partial \xi},$$

$$Q_n^{(1)} = \frac{a^2 k_n}{\gamma_n \delta_n}\left(\frac{\xi_n - \xi_{n-1}}{\eta - \xi_{n-1}}\right)^2,$$

where

$$x_n(t) = x_* \eta(\tau); \quad a^2 = \frac{\lambda_* t_*}{c_* \rho_* x_*^2}.$$

The problem in such a formulation can only be solved numerically. An effective algorithm for the numerical solution is presented below. The iteration method of L. V. Kantorovich for solving nonlinear operator equations [1] and the direct methods for solving boundary value problems for linear differential equations by reduction to Cauchy problems [2] and by the factorization method [3] are used in the algorithm.

2. Method of the Numerical Solution

Equations (1) are integrated by an implicit scheme of the form

$$Q_1 \frac{d^2 u}{d\xi^2} + Q_2 = (1 + \varkappa)\frac{u - v}{h_v} - \varkappa \frac{u - w}{h_w} \equiv D_\tau(u), \quad \varkappa = \frac{h_v}{h_w - h_v}. \tag{7}$$

The right side of this equation is the derivative $\partial u/\partial \tau\ (\xi, \tau)$ expressed in terms of the v and w distributions at the times $\tau - h_v$ and $\tau - h_w$, respectively. The parameter $\varkappa$ in (7) is formally taken as zero in the first time step. The boundary value problem for the quasilinear equation (7) with the conditions (2), (3) is solved by the L. V. Kantorovich iteration method [1] for fixed τ, which results in the following linear operator L[u] in this case:

$$L[u] = Q_1^0 \frac{d^2 u}{d\xi^2} + Q_2^0 + Q_3\left(\frac{du}{d\xi} - \frac{du^0}{d\xi}\right) + Q_4(u - u^0) - D_\tau(u), \tag{8}$$

where

$$Q_i^0 = Q_i\left(\tau, \ \xi, \ u^0, \ \frac{du^0}{d\xi}\right), \quad i = 1, \ 2;$$

$$Q_3 = \frac{\partial Q_1}{\partial p}\Big|_0 \frac{d^2u^0}{d\xi^2} + \frac{\partial Q_2}{\partial p}\Big|_0, \quad Q_4 = \frac{\partial Q_1}{\partial u}\Big|_0 \frac{d^2u^0}{d\xi^2} + \frac{\partial Q_2}{\partial u}\Big|_0, \quad p = \frac{\partial u}{\partial \xi}.$$

In each iteration the boundary value problem for the linear equation (8) with conditions (2)-(3) is solved by the method of reduction to a Cauchy problem [2] for the system:

$$\begin{aligned}
&\frac{d\varphi_1}{d\xi} = \psi_1, \quad \frac{d\psi_1}{d\xi} = \frac{1}{Q_1^0}\left[\left(\frac{1}{h_v}\cdot\frac{1+2\varkappa}{1+\varkappa} - Q_4\right)\varphi_1 - Q_3\psi_1\right],\\
&\frac{d\varphi_2}{d\xi} = \psi_2, \quad \frac{d\psi_2}{d\xi} = \frac{1}{Q_1^0}\left[\left(\frac{1}{h_v}\cdot\frac{1+2\varkappa}{1+\varkappa} - Q_4\right)\varphi_2 - Q_3\psi_2\right],\\
&\frac{d\varphi_3}{d\xi} = \psi_3, \quad \frac{d\psi_3}{d\xi} = \frac{1}{Q_1^0}\left[(1+\varkappa)\frac{\varphi_3 - v}{h_v} - \varkappa\frac{\varphi_3 - w}{h_w}\right.\\
&\qquad \left. - Q_3(\psi_3 - u_\xi^0) - Q_4(\varphi_3 - u^0) - Q_2^0\right].
\end{aligned} \tag{9}$$

The fundamental system of solutions of the system (9) could be found numerically in a standard manner and the arbitrary constants could be found from the boundary and discontinuity conditions. The exponential error accumulation during numerical integration of the system (9) by Runge—Kutta type methods is aggravated in this case because the time steps h_v and h_w in the right sides of (9) can be quite small in strongly nonstationary problems, in pulse heating say. The mentioned difficulty is successfully overcome by the following modification of the method of constructing the general solution.

The fundamental system of solutions of (9) is sought from the class of piecewise-continuous functions in $[\xi_0, \xi_n]$, which satisfy the differential equation in the intervals (ξ_0, ξ^1), (ξ^1, ξ^2), . . . , (ξ^s, ξ_n), where the points ξ_j separating the layers are included in the set $W \equiv \{\xi^1, \xi^2, \ldots, \xi^s\}$. This is done thus. Let us put $\varphi_3(\xi_0) = u^0(\xi_0, \tau)$, $\psi_3(\xi_0) = u_\xi^0(\xi_0, \tau)$ and let us select values of φ_1, φ_2, ψ_1, ψ_2 at the point ξ_0 such that the Wronskian $(\varphi_1\psi_2 - \psi_1\varphi_2)$ would not be zero. Let us integrate the system (9) numerically between ξ_0 and ξ^1 by the Runge—Kutta method, where ξ_1 is the first point at which one of the functions $|\varphi_1|$, $|\varphi_2|$, $|\varphi_3|$ becomes greater than some number N. New initial data are selected at the point ξ^1 and the procedure is duplicated for the interval (ξ^1, ξ^2), etc. The general solution of the system (9) is constructed thus, and the solution and its derivative can undergo discontinuities at the points $\xi^i \in W$ in the interval (ξ^i, ξ^{i+1}). There are 2s arbitrary constants in conformity with the number of intervals, which are selected such that conditions (2'), (3') would be satisfied, as would the conditions of continuity of u and $\partial u/\partial\xi$ at the points $\xi^k \neq \xi_j$. The linear part of the system of algebraic equations thus originating is solved by the factorization method.

3. Example

The problem of the temperature distribution in a two-layered heat shield is examined under pulsed heating conditions. It is assumed that the outer layer of the shield, which receives the thermal load V(t), can be cooled because of mass entrainment. Depending on the heat exchange conditions on the shielded surface (substrate), the quality of the heat shield is determined either by the maximal value of the substrate temperature ($\alpha = 0$) or by the quantity of absorbed heat thereon ($\alpha = \infty$). Linear relationships of ideal heat contact are taken as the discontinuity conditions for $\partial u/\partial\xi$ on the layer contact surfaces. The outer surface of the shield can be cooled by mass entrainment (for $T = T_S$), radiation and heat conduction. The thermophysical properties of the inner layer, as well as the specific heat and density of the outer layer are assumed constant, and k_2 is a linear function of T. Under the assumptions made the functions $u(\xi, \tau)$ and $\eta(\tau)$ are determined by the following equations and conditions:

$$\frac{\partial u_j}{\partial\tau} = P_j\frac{\partial^2 u_j}{\partial\xi^2} + Q_j, \quad j = 1, 2; \ \tau \in [0, \tau_1], \ \xi \in (0, \xi_1) \cup (\xi_1, 1),$$

$$(1 - s)\frac{\partial u}{\partial\xi}(0, \tau) = s[u_g - u(0, \tau)], \quad \alpha = \frac{s}{1 - s}, \quad 0 \leqslant s \leqslant 1,$$

$$\frac{\partial u^-}{\partial\xi} = \frac{k_2(u^+)}{k_1}\cdot\frac{1 - \xi_1}{\eta - \xi_1}\cdot\frac{\partial u^+}{\partial\xi}, \quad u^+ = u^-; \quad f^\pm \equiv f(\xi_1 \pm),$$

$$u_j(0, \xi) = u_j^0(\xi), \quad j = 1, 2; \ \eta(0) = 1, \quad k_2(u) = k_s + h(u - u_s),$$

for $u(1, \tau) < u_S$

$$q(\tau) = Au^4(1, \tau) + Bk_2[u(1, \tau)] \frac{1-\xi_1}{\eta-\xi_1} \cdot \frac{\partial u}{\partial \xi}(1, \tau); \quad \frac{d\eta}{d\tau} = 0$$

or

$$\frac{d\eta}{d\tau} = -\omega_0 \left[q(\tau) - Au_s^4 - Bk_s \frac{1-\xi_1}{\eta-\xi_1} \cdot \frac{\partial u}{\partial \xi}(1, \tau) \right], \quad u(1, \tau) = u_s.$$

Here

$$P_1 = \frac{a^2 k_1}{\gamma_1 \delta_1}; \quad Q_1 \equiv 0; \quad Q_2 = \frac{\partial u_2}{\partial \xi} \left[\frac{a^2 h}{\gamma_2 \delta_2} \left(\frac{1-\xi_1}{\eta-\xi_1} \right)^2 \right.$$

$$\left. \times \frac{\partial u_2}{\partial \xi} + \frac{d\eta}{d\tau} \cdot \frac{\xi-\xi_1}{\eta-\xi_1} \right];$$

$$P_2 = \frac{a^2 k_2(u_2)}{\gamma_2 \delta_2} \left(\frac{1-\xi_1}{\eta-\xi_1} \right)^2; \quad A = \frac{\varepsilon \sigma T_*^4}{V_*}; \quad B = \frac{\lambda_* T_*}{x_* V_*}; \quad V_* q(\tau) = V(t).$$

The external heat load V(t) is delivered as a triangular pulse:

$$q(\tau) = \begin{cases} 0, & 0 > \tau > 1, \\ 2\tau, & 1 \geqslant 2\tau \geqslant 0, \\ 2(1-\tau), & 2 \geqslant 2\tau \geqslant 1. \end{cases}$$

The thermophysical characteristics of the inner layer are selected in conformity with the properties of titanium in the numerical solution of the problem. The values of the parameters referring to the subliming layer correspond to the properties of silicon carbide and some typical heat-shielding materials [4-6]:

$$x_* = x_2 = 0.01 \text{ m}; \quad x_1 = 0.005; \quad \lambda_* = \lambda_1 = 0.018 \text{ kW/m} \cdot {}^\circ\text{K}, \quad \lambda_s = 0.01;$$
$$\lambda_2 = \lambda_s - 2.5 \cdot 10^{-6}(T - T_s); \quad T_* = 1000\,{}^\circ\text{K}; \quad T_s = 2800,\ 2100,\ 1400;$$
$$Q_s = 0.15 \cdot 10^{-7}, \quad 0.2 \cdot 10^{-7}, \quad 0.3 \cdot 10^{-7} \text{ kg/kW} \cdot \text{sec}; \quad \rho_* = \rho_1 = 4500 \text{ kg/m}^3;$$
$$\rho_2 = 3120; \quad c_* = c_2 = 1{,}6 \quad \text{m}^2 \cdot {}^\circ\text{K/kW}; \quad c_1 = 0{,}6; \quad \varepsilon = 1;$$
$$V_* = 10^4 \text{ kW/m}^2; \quad t_* = 30 \text{ sec.}$$

The value of V_*, as well as the duration of the pulse V(t) correspond in order of magnitude to the thermal heating modes of vehicles re-entering the dense layers of the atmosphere [7]. Examples of the results obtained are shown in Fig. 1-2. Computations correspond to the varations s = 1 and s = 0 of the heat elimination conditions on the substrate. Moreover, the values of T_s and Q_s were variated, where $T_s = 2800°K$ and $Q_s = 0.15 \cdot 10^{-7}$ kg/kW • sec were selected according to the properties of SiC_2 [6], the temperatures $T_s = 2100°K$ and $T_s = 1400°K$ are taken arbitrarily, and their corresponding Q_s values are calculated from the empirical relationship Q_sT_s = const which is valid for some materials.

Temperature profiles $T(\tilde{x}, t)$ are presented in Fig. 1a for some times t at $T_s = 2800°K$. The solid lines correspond to the s = 1 case, and the dashes to s = 0. The temperature profiles $T(\tilde{x}, t)$ for $T_s = 1400°K$ are shown in Fig. 1b. The locations $\tilde{x}_2(t)$ of the outer surface of the subliming layer are noted for several values of t in this same figure. An electronic computer computation of variations with a high sublimation rate was cut off upon compliance with the condition $\tilde{x}(t)-\tilde{x}_1 \leq 0.1\ [\tilde{x}_2(0)-\tilde{x}_1]$. The s = 1 and s = 0 versions are here denoted exactly as in Fig. 1a. Presented in Fig. 2 are time dependences of the dimensionless (referred to V_*) heat fluxes $q(\tau)$, $q_\lambda^i(\tau)$ and $q_\lambda^e(\tau)$ for various values of T_s and heat exchange conditions on the substrate (s = 0, s = 1).

In discussing the results obtained, we note that the main purpose of solving this specific problem is to illustrate the general formulation (1)-(6) of these kind of problems and their numerical solution on electronic computers.

As is seen from Figs. 1-2, the selected heat shield appears to be unsatisfactory for the pulse intensity mentioned V(t). For s = 1 the heat flux $q_\lambda^i(\tau)$ which passes within the shielded system turns out to be

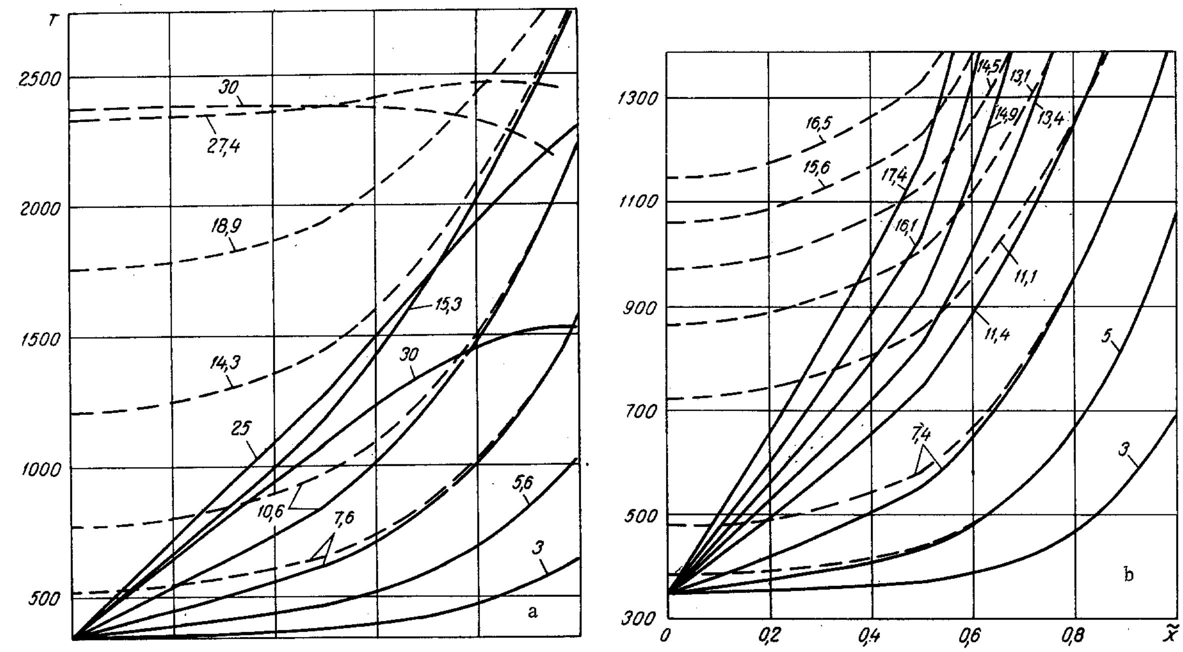

Fig. 1. Temperature distribution in layers of the system for T_s = 2800°K (*a*) and T_s = 1400°K (b). Solid lines s = 1, dashes s = 0; T in °K.

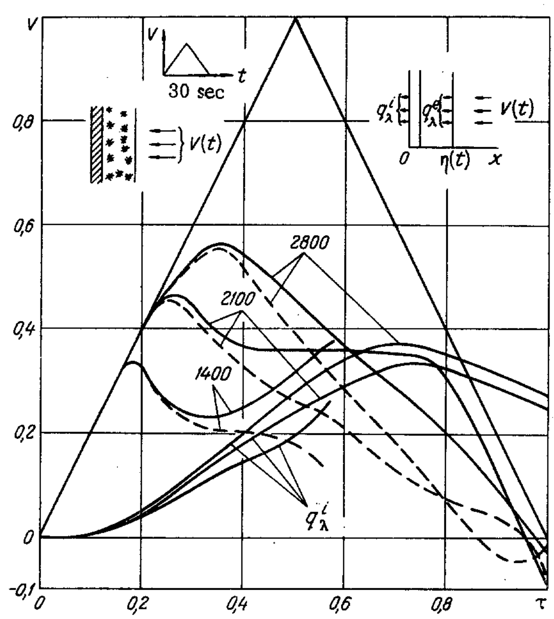

Fig. 2. Time dependence of the heat fluxes on the substrate and the surface of the subliming layer.

too great. In the case of total insulation (s = 0) the substrate temperature during the pulse duration takes on an inadmissibly high value. Evidently, the arbitrarily chosen thickness of both layers turns out to be too slight to assure satisfactory heat shielding of the substrate. It is important to note that possibilities are disclosed in the example considered, which can be used in real heat shield systems when optimal coatings are selected in some sense. One of the parameters subject to optimization is the parameter $s (0 < s < 1)$. The values of s taken in the computations correspond to the two extreme idealized cases of heat exchange on the surface x = 0 and can turn out to be necessary only under special conditions. The relationship between the layer thicknesses should also have a high value under nonstationary heating conditions. An examination of the heat shielding properties of thick subliming coatings with low values of T_s (corresponding to high entrainment rates) is of interest.

If the use of subliming coatings to shield vehicles re-entering the atmosphere [7] is hence kept in mind, for example, there arises a constraint on the weight of the heat shield and the need to select the optimal thickness of the subliming layer. Concerning the influence of the coating thermophysical characteristics (c, ρ, λ), let us just note the following. As is seen from Figs. 1a and b, very large temperature gradients can originate in the subliming layer. If the coating material possesses no great mechanical strength, this will be a constraint on the selection of the heat conduction coefficients. Let us still note here that, in particular, the maximal value of the temperature gradient in the layers is a functional of the pulse shape V(t) and a shield; optimal for one shape $V_1(t)$, is not optimal for another $V_2(t)$ even if both pulses are characterized by identical values of the maximal intensity and total energy.

As is seen, the optimality criteria of the heat shield can include any factors depending on the purpose and conditions of using the coating. Parametric computations of the temperature fields, analogous to those presented here, may turn out to be necessary in specific cases.

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